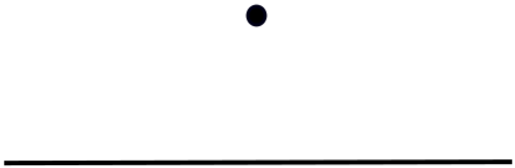
Hyperbolic Geometry and Topology

Ana Wright

September 9, 2019

Hyperbolic Geometry

Lobachevski (1829) and Bolyai (1832)
Replacing the parallel postulate of Euclidean geometry gives us a brand new geometry!



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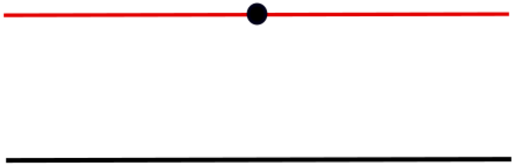
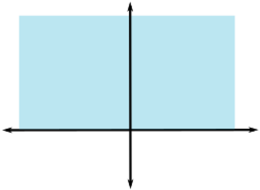
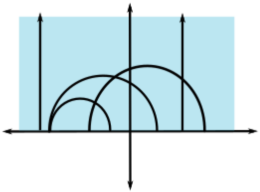


Models for Hyperbolic Space



Upper-Half Plane

Models for Hyperbolic Space



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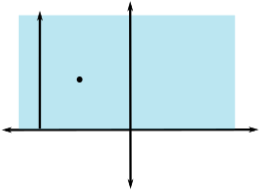

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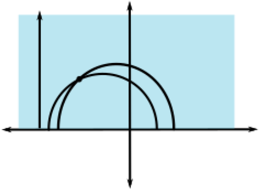

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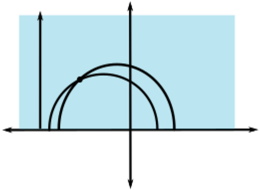

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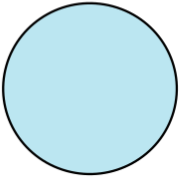
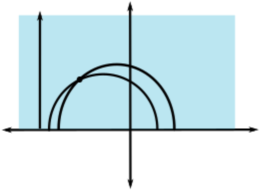


Poincaré Disk

Models for Hyperbolic Space



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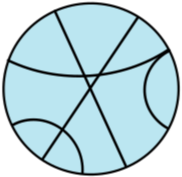

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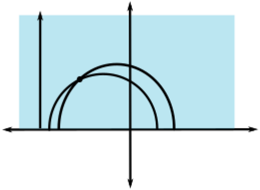

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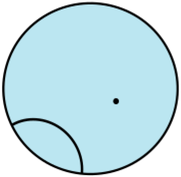

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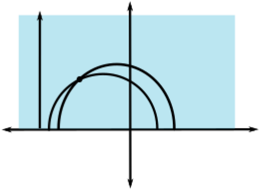

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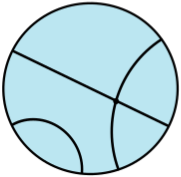

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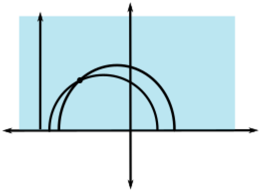

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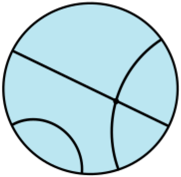

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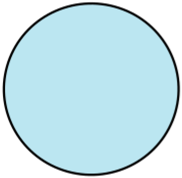

Beltrami-Klein

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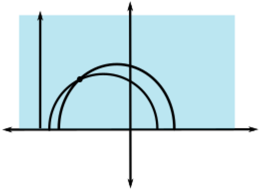

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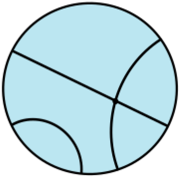

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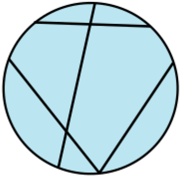

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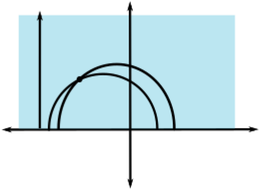

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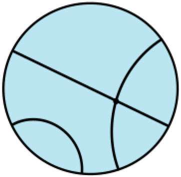

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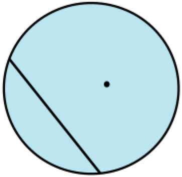

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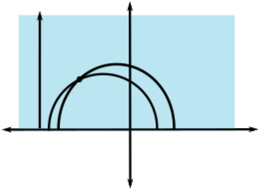

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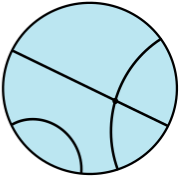

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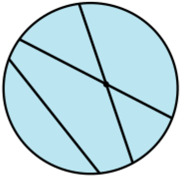

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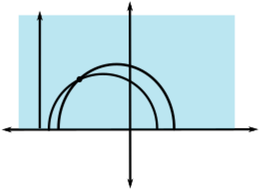

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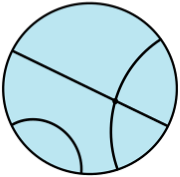

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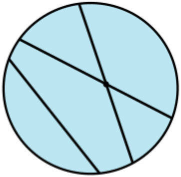


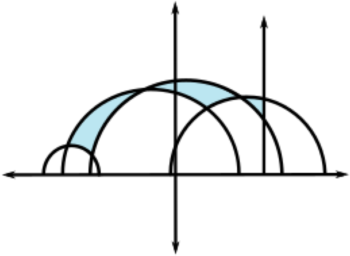
Beltrami-Klein

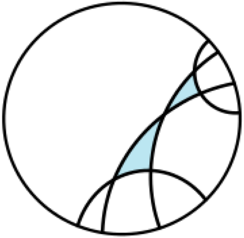
Conformal Models

Nonconformal

Getting a Feel for Hyperbolic Space

The sum of the angles of a triangle is strictly less than π.





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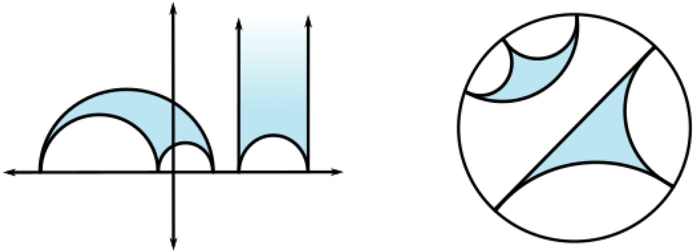
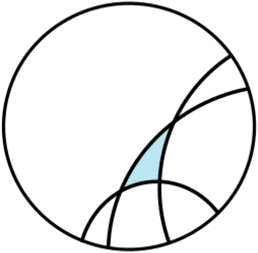


Ideal Triangles

Getting a Feel for Hyperbolic Space

AAA Congruence Theorem



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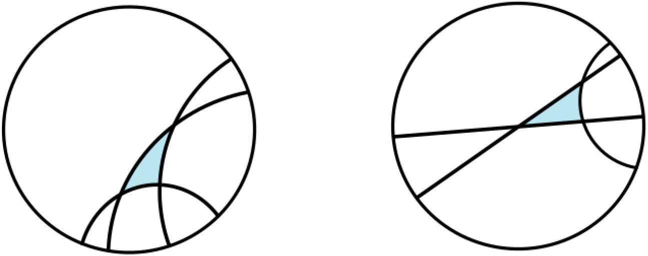

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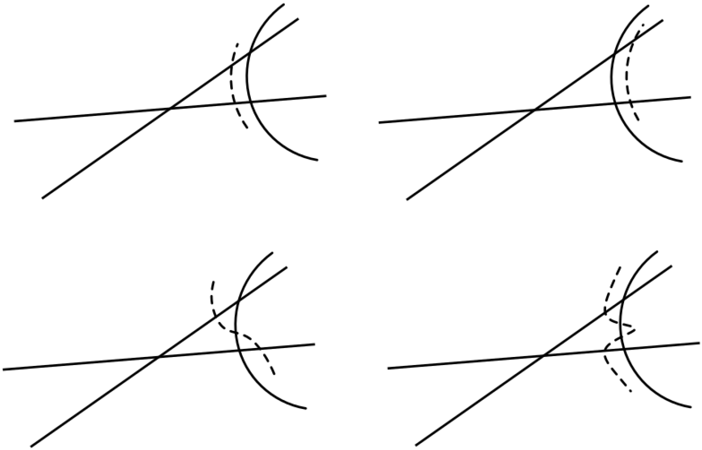

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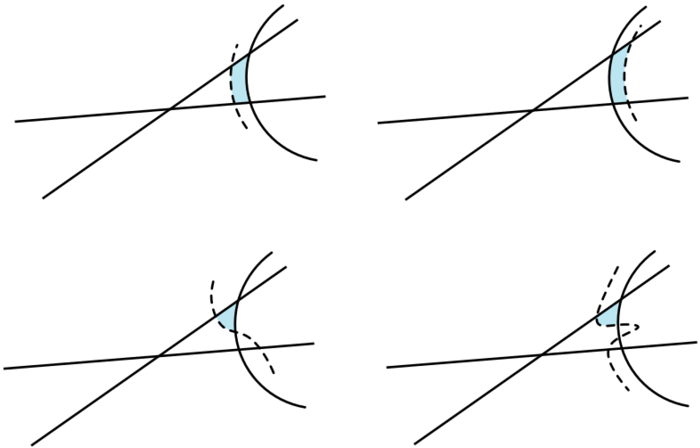
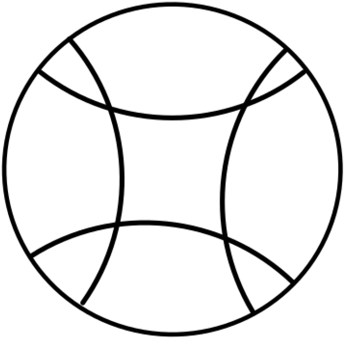


What about Topology?

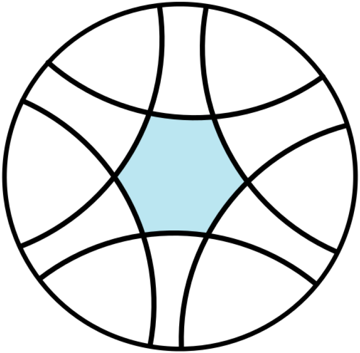
We can't put a hyperbolic structure on the torus...

Morally, this is because there are no hyperbolic rectangles.



What about Topology?

However, we do have all-right hexagons!



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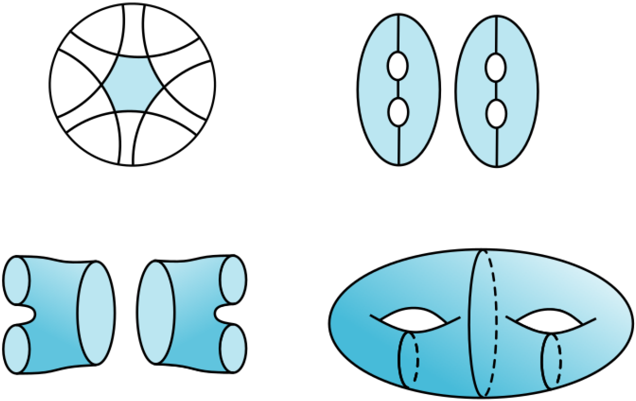

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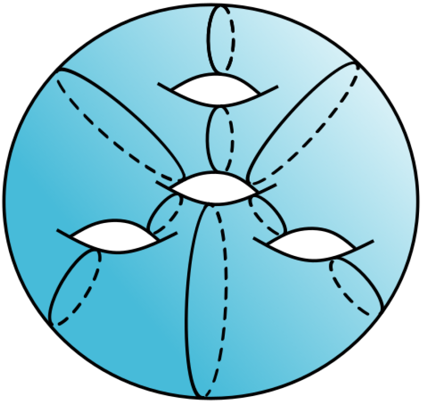
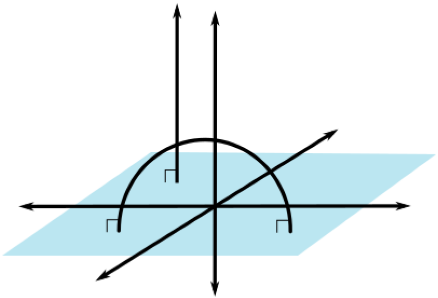
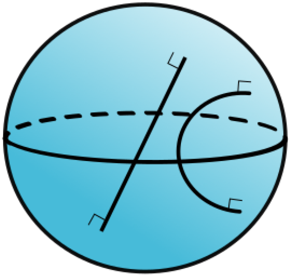


Let’s Kick it up to 3 Dimensions!



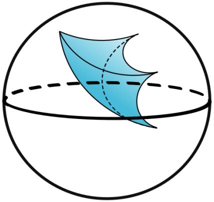
Upper-Half Space



Poincaré Ball

Let’s Do Hyperbolic Geometry on Knots!

- We can put hyperbolic structures on knot compliments in S^3 (Riley, 1973)
- We have an explicit construction! (Thurston, 1977)
- Idea: Split the knot compliment into tetrahedra with vertices on the knot and glue in ideal tetrahedra from hyperbolic space in such a way that angles work out.



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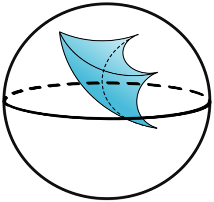

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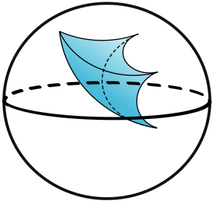

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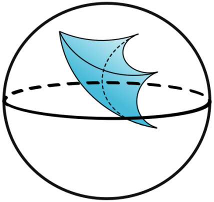
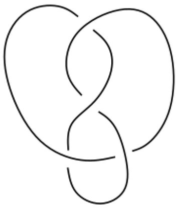


Figure-Eight Knot



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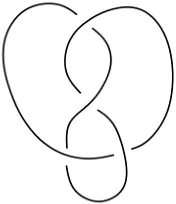

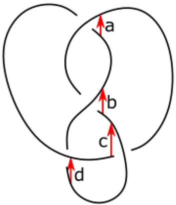

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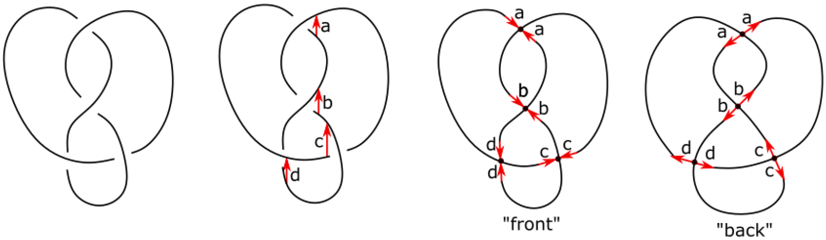

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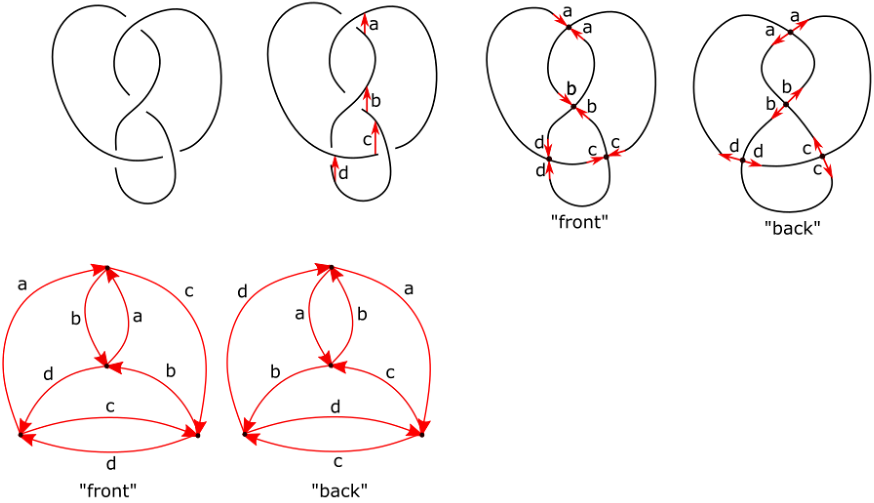

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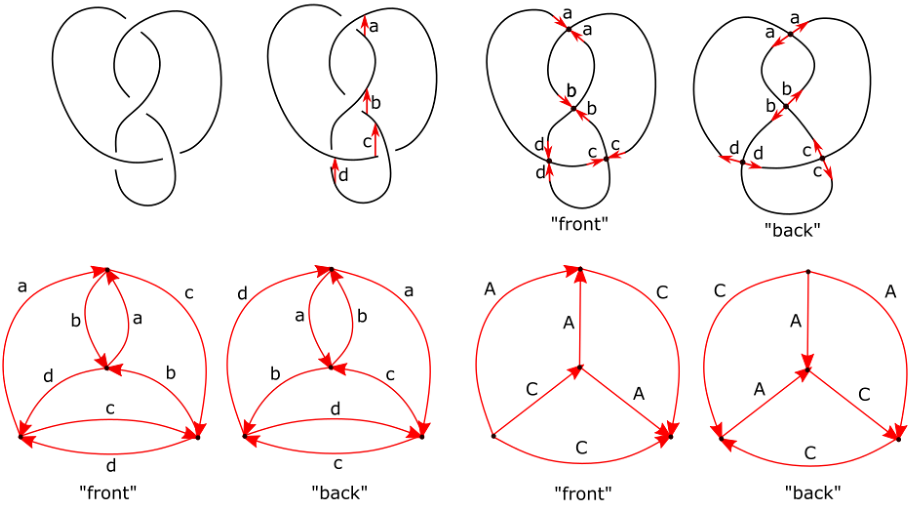

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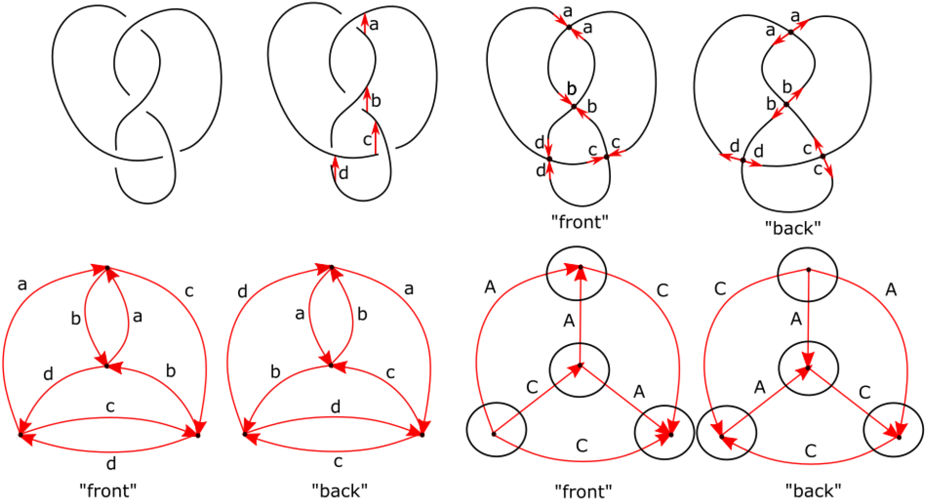

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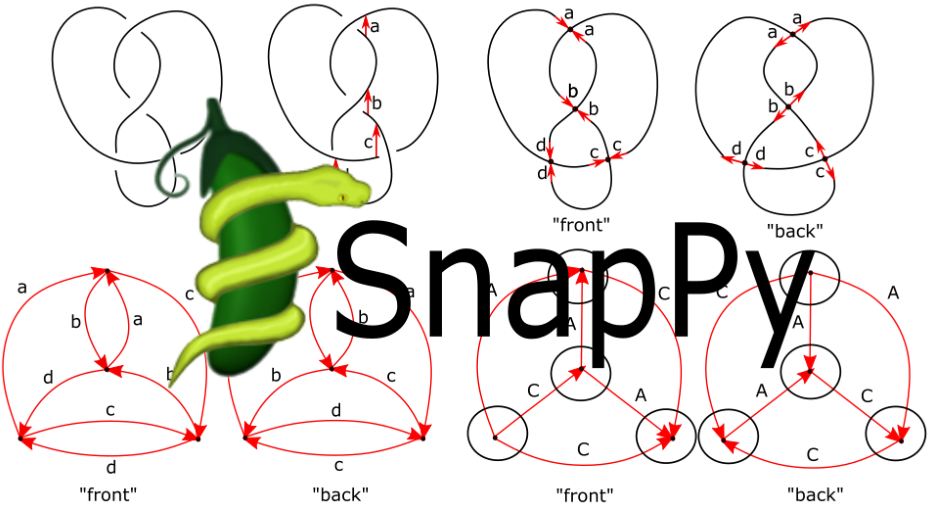

Hyperbolic Volume

- The hyperbolic volume of a knot (link) which admits a hyperbolic structure is a knot (link) invariant!
- “Most” knots admit hyperbolic structure (hyperbolic knot).
- Each hyperbolic volume has a finite number of knots with that volume.
- The set of hyperbolic volumes is well-ordered.
- It is unknown whether any hyperbolic volume is rational.
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- The figure-8 knot has the smallest hyperbolic volume of any knot: 2.02988... (Cao, Meyerhoff, 2001)
- The Whitehead link and the (-2,3,8) pretzel knot have the smallest hyperbolic volume of any 2-component link: 3.663862377... (Gabai, Meyerhoff, Milley, 2009)
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